The Disturbance in a Semi-infinite Elastic Solid due to a Linear Surface Impulse

著者	Handa Hirokichi, Nakamara Kohei, Takagi Akio
雑誌名	Science reports of the Tohoku University. Ser. 5, Geophysics
巻	8
号	1
ページ	86-92
発行年	1956-11
URL	http://hdl.handle.net/10097/44566

The Disturbance in a Semi-infinite Elastic Solid due to a Linear Surface Impulse

By HIROKICHI HONDA, KÔHEI NAKAMURA and AKIO TAKAGI

Geophysical Institute, Faculty of Science, Tôhoku University

(Received 15 May 1956)

Abstract

The problem on the propagation of the dilatational and the distortional waves through the interior of a semi-infinite elastic solid, which are generated by a normal impulse applied uniformly along a line of the surface, is investigated theoretically. The results are applied to the study of the propagation of the elastic waves of ultrasonic frequencies traveling through small scale two-dimensional models.

1 Introduction

The propagaiton of tremors over the surface of a semi-infinite elastic solid due to a vertical impulse applied uniformly along a line or at a point of the surface, was discussed by H. LAMB (1904), and that due to an internal point source by T. SAKAI (1934), and that due to a line source in the solid by H. NAKANO (1925) and E. R. LAPWOOD (1949). Similar investigations have been performed by many seismologists; T. HIRONO (1948, 1949), basing on the study carried out by NAKANO (1930), investigated the propagation of disturbances in the interior of a semi-infinite elastic solid subjected to a periodic or an aperiodic force impressed on some area of its surface.

Recently, the propagation of the elastic waves of ultrasonic frequencies traveling through the small three-dimensional models have been studied experimentally by many investigators: T.D. NORTHWOOD and D.V. ANDERSON (1953), P.N.S. O'BRIEN (1955), and others. For the discussion of the amplitude of the dilatational and distortional waves, generated by the surface traction and propagated through the medium, the results of the theoretical investigation of HIRONO will be useful. On the other hand, J. OLIVER, F. PRESS and M. EWING (1954) and Y. KATO and A. TAKAGI (1955, 1956) have performed the experimental studies for the two-dimensional models using a thin plate, as the two-dimensional models have various advantages over the three dimensional ones.

In the present paper, we study theoretically the propagation through the interior of the medium of the dilatational and distortional waves, generated by a normal impulse applied uniformly along a line of the surface of a semi-infinite elastic solid. The results are applied to the investigation of the elastic waves of ultrasonic frequencies in two-dimensional models. The velocity of the dilatational waves in a thin plate is slightly less than the usual one in an infinite solid, and it may not be difficult to give proper corrections if necessary.

2 Theory

We deal with a semi-infinite elastic solid, and take the rectangular coordinates (x, y), so that $y=0$ should conicide with the surface of the solid lying on the positive side of the plane. We suppose that the motion of the solid to be in two-dimensions. The displacement components are denoted by u and v, the density by ρ, the period of the motion by $2\pi/\omega$, the time by t, and the LAMÉ's constants by λ and μ. After LAMB, we can show that u and v produced by the normal force $Q \exp(i\omega t)$ acting parallel to the y-axis on the line $(x=0, y=0)$ in the surface, per unit length of it, can be expressed by following formulae :

$$u = u_1 + u_2 , \quad v = v_1 + v_2 , \tag{1}$$

$$u_1 = -\frac{iQ}{2\pi\mu} I_1 , \quad u_2 = \frac{iQ}{2\pi\mu} I_2 , \quad v_1 = \frac{Q}{2\pi\mu} I_3 , \quad v_2 = -\frac{Q}{2\pi\mu} I_4 , \tag{2}$$

$$\left.\begin{aligned} I_1 &= \int_{-\infty}^{\infty} \frac{\xi(2\xi^2-k^2)}{F(\xi)} \exp(-\alpha y + i\xi x)\, d\xi , \quad I_2 = \int_{-\infty}^{\infty} \frac{2\alpha\beta\xi}{F(\xi)} \exp(-\beta y + i\xi x)\, d\xi , \\ I_3 &= \int_{-\infty}^{\infty} \frac{\alpha(2\xi^2-k^2)}{F(\xi)} \exp(-\alpha y + i\xi x)\, d\xi , \quad I_4 = \int_{-\infty}^{\infty} \frac{2\alpha\xi^2}{F(\xi)} \exp(-\beta y + i\xi x)\, d\xi , \end{aligned}\right\} \tag{3}$$

$$F(\xi) = (2\xi^2 - k^2)^2 - 4\xi^2 \alpha\beta \tag{4}$$

$$h = \sqrt{\rho/(\lambda + 2\mu)} \cdot \omega = \omega/v_p , \quad k = \sqrt{\rho/\mu} \cdot \omega = \omega/v_s .$$

α and β are the positive real or positive imaginary quantities determined by

$$\alpha^2 = \xi^2 - h^2 , \quad \beta^2 = \xi^2 - k^2 , \quad \xi \text{ being real} .$$

v_p and v_s are the velocities of the dilatational and distortional waves respectively. The time factor $\exp(i\omega t)$ is here and often in the sequel temporarily omitted.

The types of the integrals are quite similar to those discussed by H. HONDA and K. NAKAMURA (1953, 1954), after SAKAI's method. No detailed exposition of the theories will be stated here, as they are fully described in these papers.

Now let us consider a complex plane defined by ζ $(\xi+i\eta)$, and represent it further upon w $(p+iq)$ plane by $\zeta=h\sin w$ for I_1 and I_3 and by $\zeta=k\sin w$ for I_2 and I_4, and confine ourselves to the region bounded by two straight lines $p=-\pi/2$ and $\pi/2$, and a certain region adjoining it if necessary.

Then we have

$$\left.\begin{aligned} I_1 &= -\int_{-\pi/2-i\infty}^{\pi/2+i\infty} \frac{\sin w \cdot \gamma\,[n^2+2(\gamma^2-1)]}{D(\gamma)} \exp[-ihr\cos(w+\theta)]\, dw , \\ I_3 &= -i\int_{-\pi/2-i\infty}^{\pi/2+i\infty} \frac{\gamma^2[n^2+2(\gamma^2-1)]}{D(\gamma)} \exp[-ihr\cos(w+\theta)]\, dw , \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} I_2 &= -2\int_{-\pi/2-i\infty}^{\pi/2+i\infty} \frac{\sin w \cdot \gamma^2\sqrt{\gamma^2-1+1/n^2}}{E(\gamma)} \exp[-ikr\cos(w+\theta)]\, dw , \\ I_4 &= 2i\int_{-\pi/2-i\infty}^{\pi/2+i\infty} \frac{\gamma(1-\gamma^2)\sqrt{\gamma^2-1+1/n^2}}{E(\gamma)} \exp[-ikr\cos(w+\theta)]\, dw , \end{aligned}\right\} \tag{6}$$

$$D(\gamma)=[n^2+2(\gamma^2-1)]^2-4\gamma(\gamma^2-1)\sqrt{\gamma^2+n^2-1}\,, \tag{7}$$

$$E(\gamma)=(2\gamma^2-1)^2-4\gamma(\gamma^2-1)\sqrt{\gamma^2-1+1/n^2}\,, \tag{8}$$

$$\gamma=\cos w,\quad n=k/h=v_p/v_s>1\,,\quad x=r\sin\theta\,,\quad y=r\cos\theta\,. \tag{9}$$

When hr and kr are large, the saddle points of the integrands of (5) and (6) lie at $w=-\theta$. The path L' of the steepest descent which passes through the saddle point ($p=-\theta$, $q=0$) is defined by $\cos(p+\theta)\cosh q=1$, and makes the angle $\pi/4$ with the real axis at the point. Putting $w+\theta=s$, $s=l\exp(i\pi/4)$ and $\cos s=1-il^2/2$ in the neighbourhood of the saddle point, and using the relation

$$\begin{aligned}\int_{L'}\exp(-ihr\cos s)\,ds &\doteqdot \int_{-\infty}^{\infty}\exp[-ihr(1-il^2/2)]\exp(i\pi/4)\,dl\\ &=\sqrt{\frac{2\pi}{hr}}\exp\left[-ihr+i\frac{\pi}{4}\right],\end{aligned} \tag{10}$$

we have

$$\left.\begin{aligned}I'_1&=\sqrt{2\pi/hr}\cdot\sin\theta\cdot G_p(\theta)\exp[-ihr+i\pi/4]\,,\\ I'_3&=-i\sqrt{2\pi/hr}\cdot\cos\theta\cdot G_p(\theta)\exp[-ihr+i\pi/4]\,.\end{aligned}\right\} \tag{11}$$

Similarly,

$$\left.\begin{aligned}I_2'&=2\sqrt{2\pi/kr}\cdot\cos\theta\cdot G_s(\theta)\exp[-ikr+i\pi/4]\,,\\ I'_4&=2i\sqrt{2\pi/kr}\cdot\sin\theta\cdot G_s(\theta)\exp[-ikr+i\pi/4]\,,\end{aligned}\right\} \tag{12}$$

$$G_p(\theta)=\cos\theta\,(n^2-2\sin^2\theta)/[(n^2-2\sin^2\theta)^2+4\sin^2\theta\cos\theta\sqrt{n^2-\sin^2\theta}]\,, \tag{13}$$

$$G_s(\theta)=\sin\theta\cos\theta\sqrt{1/n^2-\sin^2\theta}/[(2\cos^2\theta-1)^2+4\sin^2\theta\cos\theta\sqrt{1/n^2-\sin^2\theta}]\,. \tag{14}$$

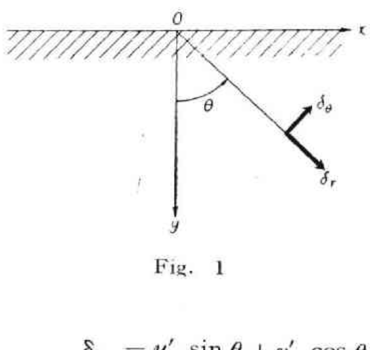

Fig. 1

The symbol (′) denotes that the quantity is due to the integral near the saddle point. I_1' and I_3' correspond to the dilatational waves and I_2' and I_4' to the distortional waves emitted from the origin. When we denote the radial component and the transverse component of displacement in the xy-plane of the dilatational waves by $\delta_{p,r}$ and $\delta_{p,\theta}$, and those of the distortional waves by $\delta_{s,r}$, and $\delta_{s,\theta}$ respectively, we have

$$\left.\begin{aligned}\delta_{p\cdot r}&=u'_1\sin\theta+v'_1\cos\theta\,, &\quad \delta_{p\cdot\theta}&=u'_1\cos\theta-v'_1\sin\theta\,,\\ \delta_{s\cdot r}&=u'_2\sin\theta+v'_2\cos\theta\,, &\quad \delta_{s\cdot\theta}&=u'_2\cos\theta-v'_2\sin\theta\,,\end{aligned}\right\} \tag{15}$$

and we have, the symbol (′) being now omitted,

$$\left.\begin{aligned}\delta_{p,r}&=\frac{\sqrt{v_p}\,Q}{\sqrt{2\pi}\,\mu}\cdot\frac{G_p(\theta)}{\sqrt{r}}\cdot\frac{1}{\sqrt{\omega}}\exp\left[i\left(\omega t-hr-\frac{\pi}{4}\right)\right],\quad \delta_{p,\theta}=0\,,\\ \delta_{s,\theta}&=-\frac{2}{\sqrt{n}}\cdot\frac{\sqrt{v_p}\,Q}{\sqrt{2\pi}\,\mu}\cdot\frac{G_s(\theta)}{\sqrt{r}}\cdot\frac{1}{\sqrt{\omega}}\exp\left[i\left(\omega t-kr-\frac{\pi}{4}\right)\right],\quad \delta_{s,r}=0\,.\end{aligned}\right\} \tag{16}$$

The integrands of (5) and (6) have the branch points corresponding to $\sqrt{\gamma^2+n^2-1}$ and $\sqrt{\gamma^2-1+1/n^2}$ and the poles corresponding to $D(\gamma)=0$ and $E(\gamma)=0$, and we must take them into consideration when θ becomes larger than certain definite values.

But as is well known, the values of the integrals around the branch cuts are proportional to $(hr)^{-3/2}$ and $(kr)^{-3/2}$ respectively, and they can be neglected when hr or kr becomes large. The integrals around the poles correspond to the surface waves of Rayleigh type, and the amplitude of which is very small when the depth from the surface is large compared with the wave lengths. When $\theta=\pi/2$, $\delta_{p,r}$ and $\delta_{s,\theta}$ of (16) vanish and require a special treatment as was pointed out by SAKAI, but the case was originally investigated by LAMB and it was shown that their amplitudes are proportional to $(hx)^{-3/2}$ or $(kx)^{-3/2}$ along the surface of the solid. Only the dilatational and distortional waves, whose amplitudes are proportional to $(hr)^{-1/2}$ and $(kr)^{-1/2}$ and predominant in the interior of the solid, are treated in the present paper.

If the time variation of the normal force applied on the line ($x=0$, $y=0$) of the surface is not simple harmonic $Q\exp(i\omega t)$, but is of the impulse type expressed by

$\Psi(t) = \dfrac{cC}{t^2+c^2}$, C, c being positive constants, we have to perform the operation

$$\frac{1}{\pi} Re\int_0^\infty d\omega\int_{-\infty}^{\infty} \psi(\sigma)\exp(-i\omega\sigma)\,d\sigma\,, \quad (Q=1) \tag{17}$$

to (16), in order to obtain the expressions $D_{p,r}$ and $D_{s,\theta}$ for the impulsive shock, which correspond to $\delta_{p,r}$ and $\delta_{s,\theta}$ for the periodic case, respectively.

Using the relation

$$\int_{-\infty}^{\infty}\frac{\exp(-i\omega\sigma)}{\sigma^2+c^2}\,d\sigma = \frac{\pi}{c}\exp(-\omega c)\,, \quad \omega>0\,,$$

we have

$$D_{p,r} = \frac{\sqrt{v_p}}{\sqrt{2}\mu}\cdot\frac{C}{\sqrt{c}}\cdot\frac{G_p(\theta)}{\sqrt{r}}\frac{1}{(1+\sigma_p{}^2)^{1/4}}\cos\left(\frac{\pi}{4}-\varphi_p\right), \tag{18}$$

$$\sigma_p = \frac{1}{c}\left(t-\frac{r}{v_p}\right), \quad \tan 2\varphi_p = \sigma_p\,.$$

For $\sin\theta < 1/n$, we have

$$D_{s,\theta} = -\frac{2}{\sqrt{n}}\frac{\sqrt{v_p}}{\sqrt{2}\,\mu}\cdot\frac{C}{\sqrt{c}}\cdot\frac{G_s(\theta)}{\sqrt{r}}\cdot\frac{1}{(1+\sigma_s{}^2)^{1/4}}\cdot\cos\left(\frac{\pi}{4}-\varphi_s\right), \tag{19}$$

$$\sigma_s = \frac{1}{c}\left(t-\frac{r}{v_s}\right), \quad \tan 2\varphi_s = \sigma_s\,.$$

For $\sin\theta > 1/n$, we have

$$D_{s,\theta} = -\frac{2}{\sqrt{n}}\cdot\frac{\sqrt{v_p}}{\sqrt{2}\,\mu}\cdot\frac{C}{\sqrt{c}}\cdot\frac{|G_s(\theta)|}{\sqrt{r}}\cdot\frac{1}{(1+\sigma_s{}^2)^{1/4}}\sin\left(\varphi_s+\varphi_s'-\frac{\pi}{4}\right), \tag{20}$$

$$\tan\varphi'_s = 4\sin^2\theta\cos\theta\sqrt{\sin^2\theta-1/n^2}/(2\cos^2\theta-1)^2\,,$$

$$|G_s(\theta)| = \sin\theta\cos\theta\sqrt{\sin^2\theta-1/n^2}/\sqrt{(2\cos^2\theta-1)^4+16\sin^4\theta\cos^2\theta(\sin^2\theta-1/n^2)}\,,$$

$$\sqrt{1/n^2-\sin^2\theta} = -i\sqrt{\sin^2\theta-1/n^2}\,.$$

3 Illustration

The wave form of $D_{p,r}$ is independent of θ, and that of $D_{s,\theta}$ also as far as $\sin\theta< 1/n$, but the latter changes with θ when $\sin\theta>1/n$. We assume that $n=\sqrt{3}$. The values of $G_p(\theta)$ and $G_s(\theta)$ have been given by HIRONO. The results of the numerical calculation for the mode of variation of $D_{p,r}$ and $D_{s,\theta}$ with time for some values of θ, together with the curves $0.805\ G_p(\theta)$ and $0.805\times 2/\sqrt{n}\ G_s(\theta)$, are illustrated in Fig. 2. The principal motion of the disturbance traveling with the

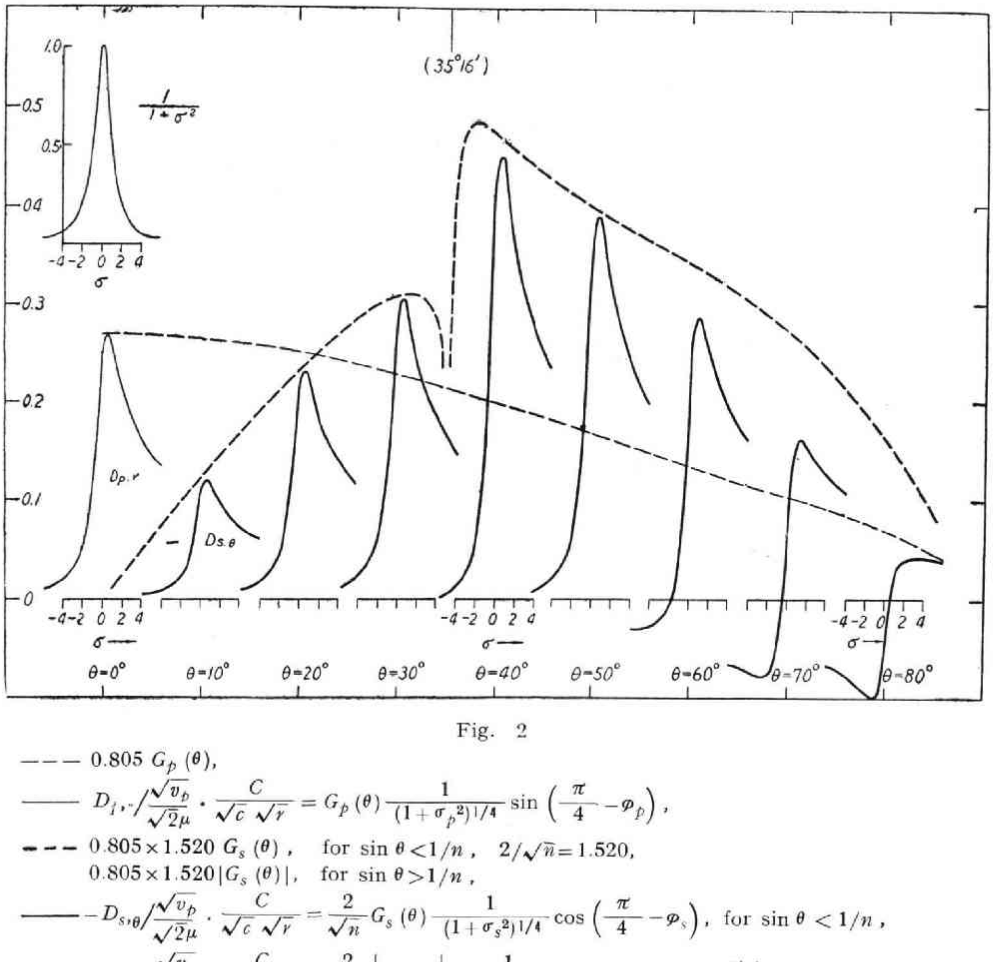

Fig. 2

--- $0.805\ G_p(\theta)$,

—— $D_{p,r}/\dfrac{\sqrt{v_p}}{\sqrt{2}\mu}\cdot\dfrac{C}{\sqrt{c}\sqrt{r}}=G_p(\theta)\dfrac{1}{(1+\sigma_p^2)^{1/4}}\sin\left(\dfrac{\pi}{4}-\varphi_p\right)$,

- - - $0.805\times1.520\ G_s(\theta)$, for $\sin\theta<1/n$, $2/\sqrt{n}=1.520$,
$0.805\times1.520|G_s(\theta)|$, for $\sin\theta>1/n$,

—— $-D_{s,\theta}/\dfrac{\sqrt{v_p}}{\sqrt{2}\mu}\cdot\dfrac{C}{\sqrt{c}\sqrt{r}}=\dfrac{2}{\sqrt{n}}G_s(\theta)\dfrac{1}{(1+\sigma_s^2)^{1/4}}\cos\left(\dfrac{\pi}{4}-\varphi_s\right)$, for $\sin\theta<1/n$,

$-D_{s,\theta}/\dfrac{\sqrt{v_p}}{\sqrt{2}\mu}\cdot\dfrac{C}{\sqrt{c}\sqrt{r}}=\dfrac{2}{\sqrt{n}}\left|G_s(\theta)\right|\dfrac{1}{(1+\sigma_s^2)^{1/4}}\sin\left(\varphi_s+\varphi_s'-\dfrac{\pi}{4}\right)$, for $\sin\theta>1/n$.

velocity of the dilatational waves is directed radially and away from the origin, and its amplitude is largest for $\theta=0$, and the principal motion of the disturbance traveling with the velocity of the distortional waves, is directed transversely to the direction of propagation and away from the surface of the solid, in general tendecny, and its amplitude is largest for about $\theta=40°$, as is shown schematically in Fig. 3.

TAKAGI (1956) investigated experimentally the propagation of the elastic waves

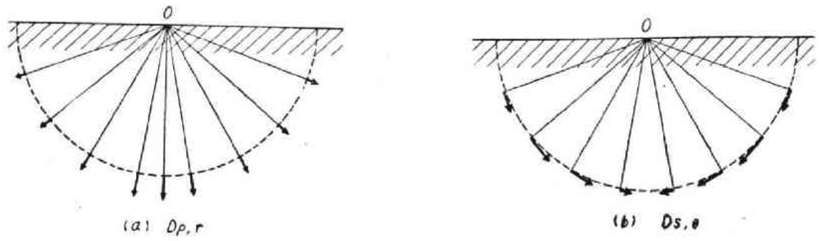

Fig. 3

of ultrasonic frequencies in the semi-circular thin bakelite plate (v_p: 3400 m/sec, v_s : 1800 m/sec) of radius 15 cm, produced by the normal shock applied at the center of the diametral side of the plate. Some of the traces of the records of the radial component ($D_{p,r}$) of the dilatational waves and the transverse component ($D_{s,\theta}$) of the distortional waves obtained at the periphery of the disk, are reproduced in Fig. 4.

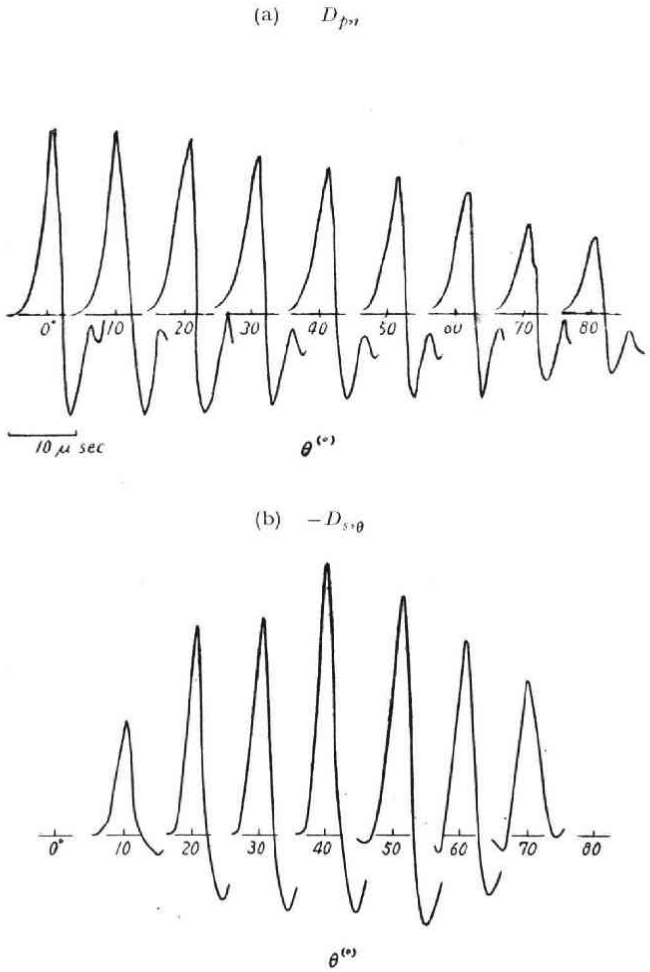

Fig. 4 The traces of the records for $D_{p,r}$ and $-D_{s,\theta}$, obtained experimentally by TAKAGI (1956).

The direction and the distribution of the amplitude of these waves shown in Fig. 4 coincide pretty well, at least in general tendency, to those obtained theoretically and illustrated in Fig. 2.

4 Summary

The propagation of the dilatational and distortional waves through the interior of a semi-infinite elastic solid, which are produced by a normal impulse applied along a line of the surface, is investigated theoretically, and the results are illustrated graphically for some special cases. The results are compared with those of the two-dimensional model experiments, and general coincidence between them is found to exist.

Reference

HIRONO, H.; Mathematical Theory on Shallow Earthquake. *Geophys. Mag.* **18**, 1-116. 1948. **21**, 1-97. 1949.

HONDA, H. and NAKAMURA, K.: Notes on the Problems on the Motion of the Surface of an Elastic Solid produced by a Linear Source. *Science Reports Tôhoku Univ. Ser.* **5**. *Geophysics*. **5**. 58-66, 1953.

On the Reflection and Refraction of the Explosive Sounds at the Ocean Bottom. *ibid.* **6**, 70-84, 1954.

KATO, Y. and TAKAGI, A.: Model Seismology. *ibid.* **7**, 35-44, 1955.

TAKAGI, A.: Seismic Model Study (Part 2) *ibid.* **8**, 69-73, 1956.

LAMB, H.: On the Propagation of Tremors over the Surface of an Elastic Solid. *Phil. Trans. Roy. Soc. London.* **A. 203**, 1-42, 1904.

LAPWOOD, E.R.: The Disturbance due to a Line Source in a Semi-infinite Elastic Medium. *ibid.* **A 841**, 63-100, 1949.

NAKANO, H.: On Rayleigh Wave. *Jap. Journ. Astro. and Geophys*, **2**, 1-94, 1925.

Some Problems concerning the Propagations of the Disturbances in and on Semi-infinite Elastic Solid. *Geophys. Mag.* **2**, 189-348, 1930.

NORTHWOOD, T.D., and ANDERSON, D.V.: Model Seismology. *Bull. Seism. Soc. Amer.* **43**, 239-245, 1953.

O'BRIEN, P.N.S.: Model Seismology-Critical Refraction of Elastic Waves. *Geophysics.* **20**, 227-242, 1955.

ORIVER, J., PRESS, F. and EWING, M.: Two-Dimensional Model Seismology. *Geophysics.* **19**, 202-219, 1954.

PRESS, F., OLVIER, J., and EWING, M.: Seismic Model Study of Refractions from a Layer of Finite Thickness. *Geophysics.* **19**, 388-401, 1954.

SAKAI, T.: On the Propagation of Tremors over the Plane Surface of an Elastic Solid produced by an Internal Source. *Geophys. Mag.* **8**, 1-71. 1934-5.